3-4 Multiplication Rule: Basics

In Section 3-3 we presented the addition rule for finding $P(A \text{ or } B)$, the probability that a trial has an outcome of A or B or both. The objective of this section is to develop a rule for finding $P(A \text{ and } B)$, the probability that event A occurs in a first trial and event B occurs in a second trial.

Notation

$P(A \text{ and } B) = P(\text{event } A \text{ occurs in a first trial and event } B \text{ occurs in a second trial})$

In Section 3-3 we associated *or* with addition; in this section we will associate *and* with multiplication. We will see that $P(A \text{ and } B)$ involves multiplication of probabilities and that we must sometimes adjust the probability of event B to reflect the outcome of event A.

Probability theory is used extensively in the analysis and design of standardized tests, such as the SAT, ACT, LSAT (for law), and MCAT (for medicine). For ease of grading, such tests typically use true/false or multiple-choice questions. Let's assume that the first question on a test is a true/false type, while the second question is a multiple-choice type with five possible answers (a, b, c, d, e). We will use the following two questions. Try them!

1. True or false: A pound of feathers is heavier than a pound of gold.
2. Among the following, which had the most influence on modern society?
   - **a.** The remote control
   - **b.** This book
   - **c.** Computers
   - **d.** Sneakers with heels that light up
   - **e.** Hostess Twinkies

The answers to the two questions are T (for "true") and c. (The first question is true. Weights of feathers are expressed in avoirdupois pounds, but weights of gold are expressed in troy pounds.) Let's find the probability that if someone makes random guesses for both answers, the first answer will be correct *and* the second answer will be correct. One way to find that probability is to list the sample space as follows:

T,a	T,b	T,c	T,d	T,e
F,a	F,b	F,c	F,d	F,e

If the answers are random guesses, then the 10 possible outcomes are equally likely, so

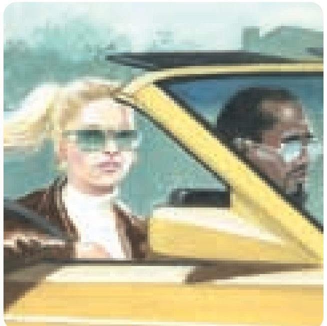
$$P(\text{both correct}) = P(T \text{ and } c) = \frac{1}{10} = 0.1$$

Now note that $P(T \text{ and } c) = 1/10$, $P(T) = 1/2$, and $P(c) = 1/5$, from which we see that

$$\frac{1}{10} = \frac{1}{2} \cdot \frac{1}{5}$$

so that

$$P(T \text{ and } c) = P(T) \times P(c)$$



Convicted by Probability

A witness described a Los Angeles robber as a Caucasian woman with blond hair in a ponytail who escaped in a yellow car driven by an African-American male with a mustache and beard. Janet and Malcolm Collins fit this description, and they were convicted based on testimony that there is only about 1 chance in 12 million that any couple would have these characteristics. It was estimated that the probability of a yellow car is 1/10, and the other probabilities were estimated to be 1/4, 1/10, 1/3, 1/10, and 1/1000. The convictions were later overturned when it was noted that no evidence was presented to support the estimated probabilities or the independence of the events. However, because the couple was not randomly selected, a serious error was made in not considering the probability of *other* couples being in the same region with the same characteristics.

This suggests that, in general, $P(A \text{ and } B) = P(A) \cdot P(B)$, but let's consider another example before making that generalization.

For now, we note that tree diagrams are sometimes helpful in determining the number of possible outcomes in a sample space. A **tree diagram** is a picture of the possible outcomes of a procedure, shown as line segments emanating from one starting point. These diagrams are helpful in counting the number of possible outcomes if the number of possibilities is not too large. The tree diagram shown in Figure 3-8 summarizes the outcomes of the true/false and multiple-choice questions. From Figure 3-8 we see that if both answers are random guesses, all 10 branches are equally likely and the probability of getting the correct pair (T, c) is 1/10. For each response to the first question, there are 5 responses to the second. The total number of outcomes is 5 taken 2 times, or 10. The tree diagram in Figure 3-8 illustrates the reason for the use of multiplication.

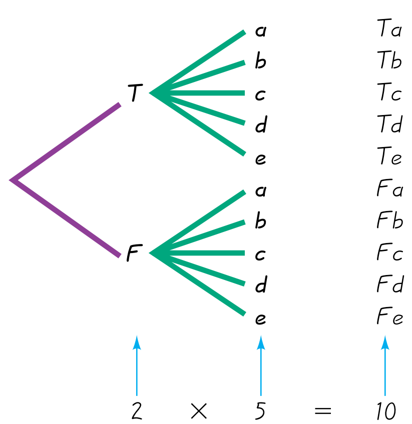


FIGURE 3-8 Tree Diagram of Test Answers

Our first example of the true/false and multiple-choice questions suggested that $P(A \text{ and } B) = P(A) \cdot P(B)$, but the next example will introduce another important element.

EXAMPLE Genetics Experiment Mendel's famous hybridization experiments involved peas, like those shown in Figure 3-3, introduced in Section 3-3 and reproduced on the next page. If two of the peas shown in Figure 3-3 are randomly selected *without replacement,* find the probability that the first selection has a green pod and the second selection has a yellow pod. (We can ignore the colors of the flowers on top.)

SOLUTION

First selection: $P(\text{green pod}) = 8/14$ (because there are 14 peas, 8 of which have green pods)

Second selection: $P(\text{yellow pod}) = 6/13$ (there are 13 peas remaining, 6 of which have yellow pods)

continued

Perfect SAT Score

If an SAT subject is randomly selected, what is the probability of getting someone with a perfect score? What is the probability of getting a perfect SAT score by guessing? These are two very different questions.

In a recent year, approximately 1.3 million people took the SAT, and only 587 of them received perfect scores of 1600, so there is a probability of 587 ÷ 1.3 million, or about 0.000452 of randomly selecting one of the test subjects and getting someone with a perfect score. Just one portion of the SAT consists of 35 multiple-choice questions, and the probability of answering all of them correct by guessing is $(1/5)^{35}$, which is so small that when written as a decimal, 24 zeros follow the decimal point before a nonzero digit appears.

With $P(\text{first pea with green pod}) = 8/14$ and $P(\text{second pea with yellow pod}) = 6/13$, we have

$$P(\text{1st pea with green pod and 2nd pea with yellow pod}) = \frac{8}{14} \cdot \frac{6}{13} \approx 0.264$$

The key point is that we must adjust the probability of the second event to reflect the outcome of the first event. Because the second pea is selected without replacement of the first pea, the second probability must take into account the result of a pea with a green pod for the first selection. After a pea with a green pod has been selected on the first trial, only 13 peas remain and 6 of them have yellow pods, so the second selection yields this: $P(\text{pea with yellow pod}) = 6/13$.

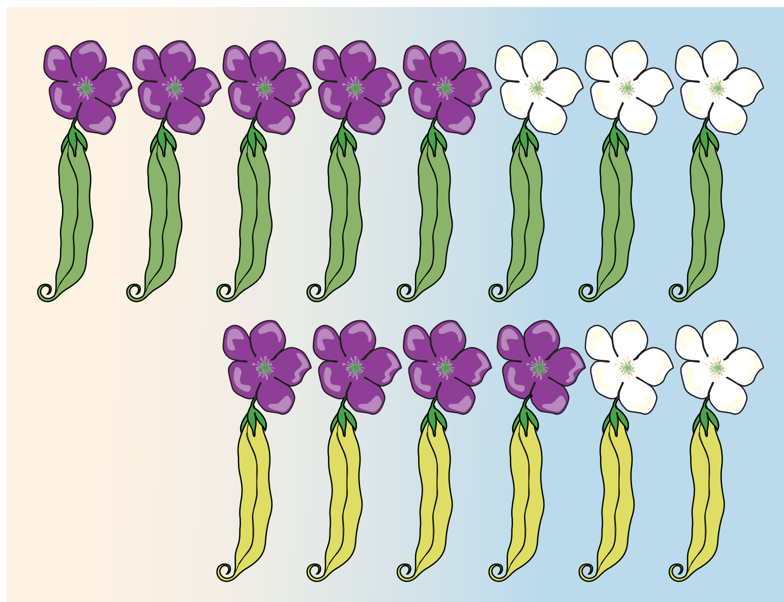


FIGURE 3-3 Peas Used in a Genetics Study

This example illustrates the important principle that *the probability for the second event* B *should take into account the fact that the first event* A *has already occurred.* This principle is often expressed using the following notation.

Notation for Conditional Probability

$P(B|A)$ represents the probability of event B occurring after it is assumed that event A has already occurred. (We can read $B|A$ as "B given A.")

Definitions

Two events A and B are **independent** if the occurrence of one does not affect the probability of the occurrence of the other. (Several events are similarly independent if the occurrence of any does not affect the probabilities of the occurrence of the others.) If A and B are not independent, they are said to be **dependent.**

STATISTICS IN THE NEWS

Lottery Advice

New York Daily News columnist Stephen Allensworth recently provided tips for selecting numbers in New York State's Daily Numbers game. In describing a winning system, he wrote that "it involves double numbers matched with cold digits. (A cold digit is one that hits once or not at all in a seven-day period.)" Allensworth proceeded to identify some specific numbers that "have an excellent chance of being drawn this week."

Allensworth assumes that some numbers are "overdue," but the selection of lottery numbers is independent of past results. The system he describes has no basis in reality and will not work. Readers who follow such poor advice are being misled and they might lose more money because they incorrectly believe that their chances of winning are better.

For example, playing the California lottery and then playing the New York lottery are *independent* events because the result of the Californian lottery has absolutely no effect on the probabilities of the outcomes of the New York lottery. In contrast, the event of having your car start and the event of getting to class on time are *dependent* events, because the outcome of trying to start your car does affect the probability of getting to class on time.

Using the preceding notation and definitions, along with the principles illustrated in the preceding examples, we can summarize the key concept of this section as the following *formal multiplication rule*, but it is recommended that you work with the *intuitive multiplication rule*, which is more likely to reflect understanding instead of blind use of a formula.

Formal Multiplication Rule

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

If A and B are independent events, $P(B \mid A)$ is really the same as $P(B)$. (For further discussion about determining whether events are independent or dependent, see the subsection "Testing for Independence" in Section 3-5. For now, try to understand the basic concept of independence and how it affects the computed probabilities.) See the following *intuitive multiplication rule*. (Also see Figure 3-9.)

Intuitive Multiplication Rule

When finding the probability that event A occurs in one trial and event B occurs in the next trial, multiply the probability of event A by the probability of event B, but be sure that the probability of event B takes into account the previous occurrence of event A.

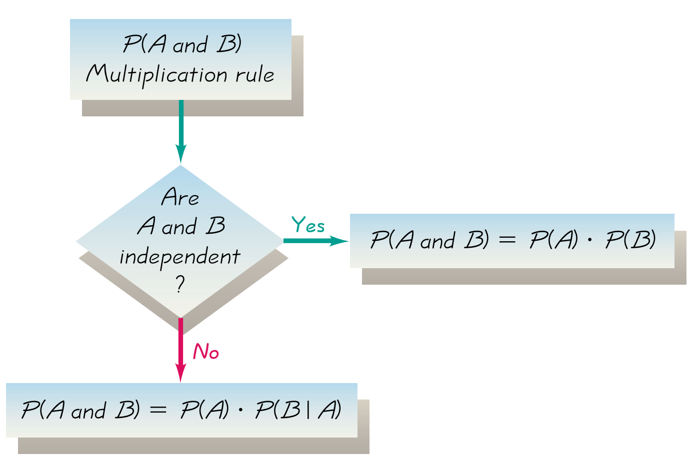


FIGURE 3-9 Applying the Multiplication Rule

Independent Jet Engines

Soon after departing from Miami, Eastern Airlines Flight 855 had one engine shut down because of a low oil pressure warning light. As the L-1011 jet turned to Miami for landing, the low pressure warning lights for the other two engines also flashed. Then an engine failed, followed by the failure of the last working engine. The jet descended without power from 13,000 ft to 4000 ft when the crew was able to restart one engine, and the 172 people on board landed safely. With independent jet engines, the probability of all three failing is only 0.0001^3, or about one chance in a trillion. The FAA found that the same mechanic who replaced the oil in all three engines failed to replace the oil plug sealing rings. The use of a single mechanic caused the operation of the engines to become dependent, a situation corrected by requiring that the engines be serviced by different mechanics.

EXAMPLE Damaged Goods Telektronics manufactures computers, televisions, CD players, and other electronics products. When shipped items are damaged, the causes of the damage are categorized as water (W), crushing (C), puncture (P), or carton marking (M). Listed below are the coded causes of five damaged items. A quality control analyst wants to randomly select two items for further investigation. Find the probability that the first selected item was damaged from crushing (C) and the second item was also damaged from crushing (C). Assume that the selections are made (a) with replacement; (b) without replacement.

W C C P M

SOLUTION

a. If the two items are selected with replacement, the two selections are independent because the second event is not affected by the first outcome. In each of the two selections there are two crushed (C) items among the five items, so we get

$$P(\text{first item is C and second item is C}) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25} \text{ or } 0.16$$

b. If the two items are selected without replacement, the two selections are dependent because the second event is affected by the first outcome. In the first selection, two of the five items were crushed (C). After selecting a crushed item on the first selection, we are left with four items including one that was crushed. We therefore get

$$P(\text{first item is C and second item is C}) = \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20} = \frac{1}{10} \text{ or } 0.1$$

Note that in this case, we adjust the second probability to take into account the selection of a crushed item (C) in the first outcome. After selecting C the first time, there would be one C among the four items that remain.

So far we have discussed two events, but the multiplication rule can be easily extended to several events. In general, the probability of any sequence of independent events is simply the product of their corresponding probabilities. For example, the probability of tossing a coin three times and getting all heads is $0.5 \cdot 0.5 \cdot 0.5 = 0.125$. We can also extend the multiplication rule so that it applies to several dependent events; simply adjust the probabilities as you go along. For example, the probability of drawing four different cards (without replacement) from a shuffled deck and getting all aces is

$$\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = 0.00000369$$

Part (b) of the last example involved selecting items without replacement, and we therefore treated the events as being dependent. However, it is a common practice to treat events as independent when *small samples* are drawn from *large populations*. In such cases, it is rare to select the same item twice. Here is a common guideline:

If a sample size is no more than 5% of the size of the population, treat the selections as being *independent* (even if the selections are made without replacement, so they are technically dependent).

Pollsters use this guideline when they survey roughly 1000 adults from a population of millions. They assume independence, even though they sample without replacement. The following example is another illustration of the above guideline. The following example also illustrates how probability can be used to test a claim made about a population. It gives us some insight into the important procedure of *hypothesis testing* that is introduced in Chapter 7.

YOU TRY IT

EXAMPLE Quality Control A production manager for Telektronics claims that her new process for manufacturing DVDs is better because the rate of defects is lower than 3%, which had been the rate of defects in the past. To support her claim, she manufactures a batch of 5000 DVDs, then randomly selects 200 of them for testing, with the result that there are no defects among the 200 selected DVDs. Assuming that the new method has the same 3% defect rate as in the past, find the probability of getting no defects among the 200 DVDs. Based on the result, is there strong evidence to support the manager's claim that her new process is better?

SOLUTION The probability of no defects is the same as the probability that all 200 DVDs are good. We therefore want to find *P*(all 200 DVDs are good). We want to assume a defect rate of 3% to see if the result of no defects could easily occur by chance with the old manufacturing process. If the defect rate is 3%, we have $P(\text{good DVD}) = 0.97$. The selected DVDs were chosen without replacement, but the sample of 200 DVDs is less than 5% of the population of 5000, so we will treat the events as if they are independent. We get this result:

$$P(\text{1st is good and 2nd is good and 3rd is good} \ldots \text{and 200th is good})$$
$$= P(\text{good DVD}) \cdot P(\text{good DVD}) \cdot \ldots \cdot P(\text{good DVD})$$
$$= 0.97 \cdot 0.97 \cdot \ldots \cdot 0.97$$
$$= 0.97^{200} = 0.00226$$

The low probability of 0.00226 indicates that instead of getting a very rare outcome with a defect rate of 3%, a more reasonable explanation is that no defects occurred because the defect rate is actually less than 3%. Because there is such a small chance (0.00226) of getting all good DVDs with a sample size of 200 and a defect rate of 3%, we do have sufficient evidence to conclude that the new method is better.

We can summarize the fundamentals of the addition and multiplication rules as follows:

- In the addition rule, the word "or" in *P*(*A* or *B*) suggests addition. Add *P*(*A*) and *P*(*B*), being careful to add in such a way that every outcome is counted only once.

STATISTICS IN THE NEWS

Redundancy

Reliability of systems can be greatly improved with redundancy of critical components. Race cars in the NASCAR Winston Cup series have two ignition systems so that if one fails, the other can be used. Airplanes have two independent electrical systems, and aircraft used for instrument flight typically have two separate radios. The following is from a *Popular Science* article about stealth aircraft: "One plane built largely of carbon fiber was the Lear Fan 2100 which had to carry two radar transponders. That's because if a single transponder failed, the plane was nearly invisible to radar." Such redundancy is an application of the multiplication rule in probability theory. If one component has a 0.001 probability of failure, the probability of two independent components both failing is only 0.000001.

- In the multiplication rule, the word "and" in $P(A \text{ and } B)$ suggests multiplication. Multiply $P(A)$ and $P(B)$, but be sure that the probability of event B takes into account the previous occurrence of event A.